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Numerical computations and theoretical investigations of a dynamical system with fractional order derivative

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Abstract This manuscript is devoted to consider population dynamical model of non-integer order to investigate the recent pandemic Covid-19 named as severe acute respiratory syndrome coronavirus-2 (SARS-CoV-2) disease. We investigate the proposed model corresponding to different values of largely effected system parameter of immigration for both susceptible and infected populations. The results for qualitative analysis are established with the help of fixed-point theory and non-linear functional analysis. Moreover, semi-analytical results, related to series solution for the considered system are investigated on applying the transform due to Laplace with Adomian polynomial and decomposition techniques. We have also applied the non-standard finite difference scheme (NSFD) for numerical solution. Finally, both the analysis are supported by graphical results at various fractional order. Both the results are comparable with each other and converging quickly at low order. The whole spectrum and the dynamical behavior for each compartment of the proposed model lying between 0 and 1 are simulated via Matlab.

1. Introduction

Since December 2019, nearly all the countries of the world have been contexting with rapidly transmitting pandemic of COVID-19. The virous of this disease was first tested in the Wuhan city of China. About 140 million population of

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humans being have infected, three millions were died and the remaining are recovered [1]. The ratio of infection, death and recovery are different in various territories of the globe. The three quantities depends on the health conditions, environmental factors, social contact, etc. Now the objective of each government and state is how to control and minimize the transmission of this effective disease. Scientist and researchers have pointed out that social contact or gathering is the main reason of this infection. Interaction of infectious one with the healthy peoples case the much more spreading of the said pandemic. From the very start of this infection up to date various strategies like lock downs on overcrowded areas, business affairs, courts hearing cases, huge buses and plane travelings, etc, have been applied. Mostly the learning institutions were effected very highly in the said lock down. Recently the law and forcemeat agencies were called to follow the SOP's on peoples by force. In short for the cycle of daily life routines some gatherings and works were allowed by the governments [2].

For all the times humans beings are trying to find better facilities for comfortable life. One of the biggest threat to human lives is the pandemic which may take their lives as well as comport. It produces tension, depression and various other diseases in the societies of human population. To find the cure for all these uncomfortable situation, different scientist and scholars along with the support of government are trying to finish it from their societies. For curing the various pandemic, they produces vaccines, medicines and also find some precautionary measures. Along all these precautions, need of fitting statistical data of the epidemic in the mathematical formula will guide about the future policy of the said disease by providing the results of infectious peoples at any time. Therefore, the importance of mathematical formulation can be seen in the investigation of different diseases or other interaction of humans or animals. This idea was fist given in 1927 by converting the real life problem into a mathematical problems. Therefore various epidemiological problems have been converted to the mathematical formula for further analysis [3–6].

The different problems in mathematical form for recent pandemic due to Corona virus have been investigated by different techniques which can be seen in [7–11]. The mathematical formulation predicted the future results of the said disease which can be then seen in the controlling process. The present and future data of this disease play important role in the construction of mathematical problem. The analysis of various mathematical modeling and their importance can be seen in [12]. Making different assumption, observation and output of the real problem will lead to the mathematical model. For this various factor can be included in form of parameters in the formula which greatly effects the diseases. The results of the formula are them compare with the real problem and will be accepted for future analysis if the are comparable with each other. The mathematical models that we used in this paper is inspired from the classic Lotka-Volterra model[13,14] for analyzing predator–prey dynamics. The classic model has been suitably modified to build the susceptible-infected individual population dynamics model. As COVID-19 can spread easily in social gathering, therefore some scholars have analyzed the dynamics of such type of disease for transmission due to immigration as given below

$$\begin{cases} \dot{\mathscr{H}} = \mathscr{H}(t)a - \mathscr{H}(t)b\mathscr{I}(t) + \mathscr{I}(t)e, \\ \dot{\mathscr{I}} = \mathscr{H}(t)b\mathscr{I}(t) + (-d - e + c)\mathscr{I}(t), \\ \mathscr{H}(0) = \mathscr{H}_0, \quad \mathscr{I}(0) = \mathscr{I}_0, \end{cases} \tag{1}$$

where the susceptible individual population is given by $\mathscr{H}(t)$ at time t, infected individual population is given by $\mathscr{I}(t)$ at time t. The infection rate is given by $b = (1\text{–protection rate})$. The immigration rate of susceptible individuals is given by a. The immigration rate of infected individuals is given by c. The death rate is given by d and the recovered rate is given by e. Further we include δ being natural death rate and third equation of recovered individuals to make another model. This model is just an indication to see what happen in a community, if immigration of individuals does not control in it.

Applying the aforementioned points, we are going to study the model given in (2) by including recovered individuals equation for fractional-order derivative with $0 < \sigma \leqslant 1$ as given by

$$\begin{cases} \mathscr{D}^{\sigma}\mathscr{H}(t) = \mathscr{H}(t)a - \mathscr{H}(t)b\mathscr{I}(t) + (-\delta)\mathscr{H}(t), \\ \mathscr{D}^{\sigma}\mathscr{I}(t) = \mathscr{H}(t)b\mathscr{I}(t) + (-d - e + c - \delta)\mathscr{I}(t), \\ \mathscr{D}^{\sigma}\mathscr{R}(t) = (e)\mathscr{I}(t) - \delta\mathscr{R}(t), \\ \mathscr{H}(0) = \mathscr{H}_0 > 0, \quad \mathscr{I}(0) = \mathscr{I}_0, \ \mathscr{R}(0) = \mathscr{R}_0 \end{cases} \tag{2}$$

where $\mathscr{H}_0$ and $\mathscr{I}_0, \mathscr{R}_0$ are starting values for susceptible, infected and recovered classes as population densities in percents. We are investigating the Eq. (2) in arbitrary order derivative of Caputo sense because it gives much well result than integer order. providing such type of results makes fractional calculus superior than integer order calculus. Therefore, the physical and biological problems may be investigated with the general fractional derivative operators on every very small order having more degree of choice. The researchers interested to work on fractional derivatives as compared to the classical differentiation. So for the analysis are composed of theory of existence and uniqueness, stability, feasibility and approximate solution of fractional order equations [13,15–20].

Fractional order calculus have provided the information of the whole spectrum lying between any two different integer values [21–24]. The representation of various real problems have been modeled by fractional order differential or integral equation like, mathematical fractional order model for microorganism population, logistic non-linear model for human population, tuberculosis model, dingy problem, hepatitis B, C models and the basic Lotka-Volterra models being the basics of all infectious problems [25–34]. The afore said problems have been analyzed for qualitative analysis with help of some well known theorems of fixed point theory [35–39]. The feasibility and stability analysis have also been done through various theorems. Further the FODEs have been checked for analytical, semi-analytical and approximate solution using different techniques. Some of the well known techniques were of Euler, Taylor, Adams–Bashforth, predictor–corrector, various transforms, etc [13,26,40–44]. Here we are going to analyze our proposed fraction order model in sense of Caputo derivative for existence, uniqueness of solution and approximate solution by theory of fixed point and Laplace transform along with some decomposition techniques. We also find the numerical solution of the proposed problem by Non-Standard Finite Difference Scheme (NSFD). We will also investigate the effect of immigration on the pandemic transmission in the society using different fractional or arbitrary orders for the considered model.

2. Basics of modern calculus

Now, we present famous definitions as in [13,15,26].

Definition 2.1. Let we have an operator $\mathscr{H}(t)$ to give the definition of non-integer order of integration w.r.t t in Caputo sense as

$$ {}^{c}{}_{0}\mathscr{I}_{t}^{\sigma}\mathscr{H}(t) = \frac{1}{\Gamma(\sigma)}\int_0^t (t-\eta)^{\sigma-1} d\eta,\ \theta > 0, $$

having satisfy the condition of convergency.

Definition 2.2. Let we have a mapping $\mathscr{H}(t)$, to provide the definition fractional order differentiation w.r.t t as

$$ {}^{c}{}_{0}\mathscr{D}_{t}^{\sigma}\mathscr{H}(t) = \frac{1}{\Gamma(n-\sigma)}\int_0^t (t-\eta)^{n-\sigma-1}[\frac{d^n}{d\eta^n}\mathscr{H}(\eta)]d\eta,\ \sigma > 0, $$

with right side quantity is piece wise continuous on $\mathbb{R}^+$ and $n = [\sigma]+1$. If $0 < \sigma \leq 1$, then we can write

$$ {}^{c}{}_{0}\mathscr{D}_{t}^{\sigma}\mathscr{H}(t) = \frac{1}{\Gamma(1-\sigma)}\int_0^t (t-\eta)^{-\sigma}[\frac{d}{d\eta}\mathscr{H}(\eta)]d\eta. $$

Lemma 2.2.1. [13] *The solution of*

$$ {}^{c}{}_{0}\mathscr{D}_{t}^{\sigma}\mathscr{H}(t) = h(t),\quad 0 < \sigma \leqslant 1 $$

is

$$ \mathscr{H}(t) = c_0 + \frac{1}{\Gamma(1-\sigma)}\int_0^t h(\eta)(t-\eta)^{\sigma-1}d\eta. $$

Lemma 2.2.2. [13]*The Laplace transform of* ${}^{c}{}_{0}\mathscr{D}_{t}^{\sigma}\mathscr{H}(t)$ *for* $0 < \sigma \leq 1$ *is provided by*

$$ \mathscr{L}\left[{}^{c}{}_{0}\mathscr{D}_{t}^{\sigma}\mathscr{H}(t)\right] = s^{\sigma}\mathscr{L}[\mathscr{H}(t)] - s^{\sigma-1}\mathscr{H}(0)]. $$

Lemma 2.2.3. *The feasible and bounded solution of the proposed system (2) is*

$$ \mathbb{S} = \left\{(\mathscr{H},\mathscr{I},\mathscr{R}) \in \mathbf{R}_+^3 :\ 0 \leqslant N(t) \leqslant \frac{a+c}{\delta}\right\}. $$

Proof. For derivation we have to add all the three equations of (2), as

$$ \begin{aligned} &\tfrac{dN}{dt} = (a)\mathscr{H}(t) - (\mathscr{H}(t)+\mathscr{I}(t)+\mathscr{R}(t))\delta + \mathscr{I}(t)c \\ &-(e+d)\mathscr{I}(t) + (e)\mathscr{I}(t) \\ &\leqslant (a)\mathscr{H}(t) + (-\delta)N(t) + c\mathscr{I}(t) \\ &= a + c - (\delta)N(t). \end{aligned} \tag{3} $$

Henceonehas $\frac{dN}{dt} + (\delta)N(t) \leqslant a + c.$

Upon integration of (3) we obtain

$$ N(t) \leqslant \frac{a+c}{\delta} + C\exp(-(\delta)t), \tag{4} $$

Here C is constant due to integration. In (4), as t tends to ∞, then

$$ N(t) \leqslant \frac{a+c}{\delta}. $$

The last result proved the boundedness and feasibility of solution. $\square$

3. Main work

In this section of the paper we will check our consider model (2) of fractional order for qualitative properties. These kinds of properties can be easily checked by fixed point theory. We will also uses some basic theorems like Banach contraction and Schauder theorems to receive our required result. Take $\mathscr{U} = \mathscr{H}(t) + \mathscr{I}(t) + \mathscr{R}(t)$.

$$ \begin{aligned} {}^{c}{}_{0}\mathscr{D}_{t}^{\sigma}\mathscr{U}(t) &= \mathscr{F}(t,\mathscr{U}(t)),\ \sigma \in (0,1], \\ \mathscr{U}(0) &= \mathscr{U}_0. \end{aligned} \tag{5} $$

Applying fractional integration of order $\sigma \in (0,1]$ to (5), we obtain

$$ \mathscr{U}(t) = \mathscr{U}_0 + \frac{1}{\Gamma(\sigma)}\int_0^t (t-\eta)^{\sigma-1}\mathscr{F}(\eta,\mathscr{U}(\eta))d\eta. \tag{6} $$

Consider $\infty > T \geqslant t > 0$ with complete norm space as $\mathscr{E}_1 = \mathscr{C}[0,T]$ under the norm

$$ \|\mathscr{U}\| = \max_{t\in[0,T]}|\mathscr{U}(t)|. $$

For qualitative properties, we take the condition of growth on non-linear operator as:

(A1) $\exists\ L_{\mathscr{F}} > 0; \forall\ \mathscr{U},\ \bar{\mathscr{U}} \in \mathscr{E}$ we have

$$ |\mathscr{F}(t,\mathscr{U}) - \mathscr{F}(t,\bar{\mathscr{U}})| \leqslant L_{\mathscr{F}}|\mathscr{U} - \bar{\mathscr{U}}|. $$

(A2) $\exists\ C_{\mathscr{F}} > 0\ \&\ M_{\mathscr{F}} > 0,;$

$$ |\mathscr{F}(t,\mathscr{U}(t))| \leqslant C_{\mathscr{F}}|\mathscr{U}| + M_{\mathscr{F}}. $$

Theorem 3.1. *Let* $\mathscr{F}$ *be continuous and satisfying* $(A2)$, *model (5) has* $\geqslant 1$ *solution.*

Proof. Applying "Schauder fixed point theorem", let us define a closed subset $\mathscr{B}$ of $\mathscr{E}$ as

$$ \mathbf{B} = \{\mathscr{U} \in \mathscr{E} :\ \|\mathscr{U}\| \leqslant \mathbf{R},\ \mathbf{R} > 0\}, $$

where

$$ \mathbf{R} \geqslant \max\left[\frac{|\mathscr{U}_0|\Gamma(\sigma+1) + T^{\sigma}M_{\mathscr{F}}}{\Gamma(\sigma+1) - T^{\sigma}C_{\mathscr{F}}}\right]. $$

Let we provide a mapping $\mathbf{T} : \mathbf{B} \to \mathbf{B}$ and using (6) as

$$ \mathbf{T}(\mathscr{U}) = \mathscr{U}_0(t) + \frac{1}{\Gamma(\sigma)}\int_0^t (t-\eta)^{\sigma-1}\mathscr{F}(\eta,\mathscr{U}(\eta))d\eta. \tag{7} $$

At any $\mathscr{U} \in \mathbf{B}$, we get

$$ \begin{aligned} |\mathbf{T}(\mathscr{U})(t)| \leqslant&\ |\mathscr{U}_0| + \tfrac{1}{\Gamma(\sigma)}\textstyle\int_0^t (t-\eta)^{\sigma-1}|\mathscr{F}(\eta,\mathscr{U}(\eta))|d\eta, \\ \leqslant&\ |\mathscr{U}_0| + \tfrac{1}{\Gamma(\sigma)}\textstyle\int_0^t (t-\eta)^{\sigma-1}[C_{\mathscr{F}}|\mathscr{U}| + M_{\mathscr{F}}]ds, \\ \leqslant&\ |\mathscr{U}_0| + \tfrac{T^{\sigma}}{\Gamma(\sigma+1)}[C_{\mathscr{F}}\|\mathscr{U}\| + M_f], \end{aligned} $$

which implies that

$$\begin{aligned}\|\mathbf{T}(\mathscr{U})\| \leqslant &\ |\mathscr{U}_0| + \tfrac{T^\sigma}{\Gamma(\sigma+1)}[C_{\mathscr{F}}\|\mathscr{U}\| + M_f],\\ \leqslant &\ \mathbf{R}.\end{aligned} \tag{8}$$

From (8), it implies that $\mathscr{U} \in \mathbf{B}$. Thus $\mathbf{T}(\mathbf{B}) \subset \mathbf{B}$. This also proves that $\mathbf{T}$ is closed. Next for completely continuous we go ahead as:

Take $t_1 < t_2 \in [0, T]$, assume

$$\begin{aligned}|\mathbf{T}(\mathscr{U})(t_2) - \mathbf{T}(\mathscr{U})(t_1)| = &\ \Big|\tfrac{1}{\Gamma(\sigma)}\int_0^{t_2}(t_2-\eta)^{\sigma-1}\mathscr{F}(s,\mathscr{U}(\eta))d\eta\\ - &\ \tfrac{1}{\Gamma(\sigma)}\int_0^{t_1}(t_1-\eta)^{\sigma-1}\mathscr{F}(\eta,\mathscr{U}(\eta))d\eta\Big|,\\ \leqslant &\ \tfrac{1}{\Gamma(\sigma)}\int_0^{t_1}[(t_1-\eta)^{\sigma-1}-(t_2-\eta)^{\sigma-1}]|\\ &\ \mathscr{F}(s,\mathscr{U}(\eta))|d\eta\\ + &\ \tfrac{1}{\Gamma(\sigma)}\int_{t_1}^{t_2}(t_2-\eta)^{\sigma-1}|\mathscr{F}(s,\mathscr{U}(\eta))|d\eta,\\ \leqslant &\ \tfrac{1}{\Gamma(\sigma)}\Big[\int_0^{t_1}[(t_1-\eta)^{\sigma-1}-(t_2-\eta)^{\sigma-1}]d\eta\\ + &\ \int_{t_1}^{t_2}(t_2-\eta)^{\sigma-1}d\eta\Big](C_{\mathscr{F}}\mathbf{R}+M_{\mathscr{F}}),\\ \leqslant &\ \tfrac{(C_{\mathscr{F}}\mathbf{R}+M_{\mathscr{F}})}{\Gamma(\sigma+1)}[t_2^\sigma - t_1^\sigma + 2(t_2-t_1)^\sigma].\end{aligned} \tag{9}$$

Now from (9), we get $t_1 \to t_2$, then the right side will approach to 0. So we say that

$$|\mathbf{T}(\mathscr{U})(t_2) - \mathbf{T}(\mathscr{U})(t_1)| \to 0,\ \text{as}\ t_1 \to t_2.$$

Consequently we claim that

$$\|\mathbf{T}(\mathscr{U})(t_2) - \mathbf{T}(\mathscr{U})(t_1)\| \to 0,\ \text{as}\ t_1 \to t_2.$$

Hence $\mathbf{T}$ is equi- continues operator. Applying Arzelá-Ascoli theorem, mapping $\mathbf{T}$ is completely continuous and which is also uniform continuous and bounded. Now by Schauder's fixed point theorem model (2) has $\geqslant 1$ solution.

Further for uniqueness we proceeds as:

Theorem 3.2. *With* $(A1)$, *the considered model has one solution if* $1 > \frac{T^\sigma}{\Gamma(\sigma+1)}L_{\mathscr{F}}$.

Proof. We have already define an operator as $\mathbf{T} : \mathbf{B} \to \mathbf{B}$, consider $\mathscr{U}$ and $\bar{\mathscr{U}} \in \mathbf{B}$ as

$$\begin{aligned}\|\mathbf{T}(\mathscr{U}) - \mathbf{T}(\bar{\mathscr{U}})\| = &\max_{t\in[0,T]}\Big|\tfrac{1}{\Gamma(\sigma)}\int_0^t(t-\eta)^{\sigma-1}\mathscr{F}(\eta,\mathscr{U}(\eta))d\eta\\ &-\tfrac{1}{\Gamma(\sigma)}\int_0^t(t-\eta)^{\sigma-1}\mathscr{F}(\eta,\bar{\mathscr{U}}(\eta))d\eta\Big|,\\ &\leqslant \tfrac{T^\sigma}{\Gamma(\sigma+1)}L_{\mathscr{F}}\|\mathscr{U}-\bar{\mathscr{U}}\|.\end{aligned} \tag{10}$$

From (10), we have

$$\|\mathbf{T}(\mathscr{U}) - \mathbf{T}(\bar{\mathscr{U}})\| \leqslant \frac{T^\sigma}{\Gamma(\sigma+1)}L_{\mathscr{F}}\|\mathscr{U}-\bar{\mathscr{U}}\|. \tag{11}$$

Hence F is contraction. So finally by the results of theorem of Banach contraction the considered problem has one solution. □

4. General series solution of the considered System (2)

To produce semi-analytical solution to the considered model, consider a general problem as

$$\begin{aligned}{}^c_0\mathrm{D}_t^\sigma\mathscr{U}(t) = &\ \mathscr{N}(\mathscr{U}(t)) + L(\mathscr{U}(t)),\quad 0 < \sigma \leqslant 1,\\ \mathscr{U}(0) = &\ \mathscr{U}_0,\ \mathscr{U}_0 \in \mathbb{R}^+,\end{aligned} \tag{12}$$

where

$$\mathscr{U}(t) = \begin{cases}\mathscr{H}(t)\\ \mathscr{I}(t)\\ \mathscr{R}(t),\end{cases} \mathscr{U}_0(t) = \begin{cases}\mathscr{H}_0(t)\\ \mathscr{I}_0(t)\\ \mathscr{R}_0(t),\end{cases} \tag{13}$$

$\mathscr{N}(\mathscr{U})$ is nonlinear term and $L(\mathscr{U})$ represent linear term. Taking Laplace transform of (12) and using initial condition, we have

$$\mathscr{L}[\mathscr{U}(t)] = \frac{1}{s}\mathscr{U}_0 + \frac{1}{s^\sigma}\mathscr{L}[\mathscr{N}(\mathscr{U}(t)) + L(\mathscr{U}(t))]. \tag{14}$$

Let us consider the required solution as $\mathscr{U}(t) = \sum_{n=0}^\infty \mathscr{U}_n(t)$ and the nonlinear term $\mathscr{N}(\mathscr{U})$ may be expressed as $\mathscr{N}(\mathscr{U}) = \sum_{n=0}^\infty \mathscr{P}_n$, where $\mathscr{P}_n$ is defined by

$$\mathscr{P}_n = \frac{1}{\Gamma(n+1)}\frac{d^n}{d\xi^n}\left[\mathscr{N}\left(\sum_{k=0}^n \xi^k\mathscr{U}_k\right)\right]\Bigg|_{\xi=0}.$$

Therefore (14) becomes

$$\mathscr{L}\left[\sum_{n=0}^\infty \mathscr{U}_n(t)\right] = \frac{1}{s}\mathscr{U}_0 + \frac{1}{s^\sigma}\mathscr{L}\left[\sum_{n=0}^\infty \mathscr{P}_n(t) + L\left(\sum_{n=0}^\infty \mathscr{U}_n(t)\right)\right].$$

On comparing, we have

$$\begin{aligned}\mathscr{L}[\mathscr{U}_0(t)] = &\ \tfrac{1}{s}\mathscr{U}_0,\\ \mathscr{L}[\mathscr{U}_1(t)] = &\ \tfrac{1}{s^\sigma}\mathscr{L}[\mathscr{P}_0 + L(\mathscr{U}_0)],\\ \mathscr{L}[\mathscr{U}_2(t)] = &\ \tfrac{1}{s^\sigma}\mathscr{L}[\mathscr{P}_1 + L(\mathscr{U}_1)],\\ &\vdots\\ \mathscr{L}[\mathscr{U}_{n+1}(t)] = &\ \tfrac{1}{s^\sigma}\mathscr{L}[\mathscr{P}_n + L(\mathscr{U}_n)].\end{aligned}$$

Evaluating inverse Laplace transform, we have

$$\begin{aligned}\mathscr{U}_0(t) = &\ \mathscr{U}_0,\\ \mathscr{U}_1(t) = &\ \mathscr{L}^{-1}\left[\tfrac{1}{s^\sigma}\mathscr{L}[\mathscr{P}_0 + L(\mathscr{U}_0)]\right],\\ \mathscr{U}_2(t) = &\ \mathscr{L}^{-1}\left[\tfrac{1}{s^\sigma}\mathscr{L}[\mathscr{P}_1 + L(\mathscr{U}_1)]\right],\\ &\vdots\\ \mathscr{U}_{n+1}(t) = &\ \mathscr{L}^{-1}\left[\tfrac{1}{s^\sigma}\mathscr{L}[\mathscr{P}_n + L(\mathscr{U}_n)]\right],\ n \geqslant 0.\end{aligned}$$

Hence the required series solution of the considered model (12) will be received as

$$\mathscr{U}(t) = \mathscr{U}_0(t) + \mathscr{U}_1(t) + \mathscr{U}_2(t)\ldots \tag{15}$$

Remark 1. The convergence of the series given in (13) has been proved in [30].

5. Approximate solution and discussion for (1)

To compute the approximate results, we take some values for parameters of (1) under fractional order. Let us in community the susceptible population be $\mathscr{H}_0$, infected be $\mathscr{I}_0$, recovered be $\mathscr{R}_0$ and taking various values for rates we discuss certain cases as bellow:

$$\begin{cases} \mathscr{D}^{\sigma}\mathscr{H}(t) = a\mathscr{H}(t) - b\mathscr{H}(t)\mathscr{I}(t) + (-\delta)\mathscr{H}(t), \\ \mathscr{D}^{\sigma}\mathscr{I}(t) = b\mathscr{H}(t)\mathscr{I}(t) + (c - d - e - \delta)\mathscr{I}(t), \\ \mathscr{D}^{\sigma}\mathscr{R}(t) = (e)\mathscr{I}(t) - \delta\mathscr{R}(t), \\ \mathscr{H}(0) = \mathscr{H}_0 > 0, \quad \mathscr{I}(0) = \mathscr{I}_0, \; \mathscr{R}(0) = \mathscr{R}_0 \end{cases} \tag{16}$$

using the proposed algorithm to (16) as constructed in (15), analogously one has

$$\begin{cases} \mathscr{H}_0(t) = \mathscr{H}_0, \quad \mathscr{I}_0(t) = \mathscr{I}_0, \quad \mathscr{R}_0(t) = \mathscr{R}_0, \\ \mathscr{H}_1(t) = [a\mathscr{H}_0 - b\mathscr{H}_0\mathscr{I}_0 + (-\delta)\mathscr{H}_0]\frac{t^{\sigma}}{\Gamma(\sigma+1)}, \\ \mathscr{I}_1(t) = [b\mathscr{H}_0\mathscr{I}_0 + (c - d - e - \delta)\mathscr{I}_0]\frac{t^{\sigma}}{\Gamma(\sigma+1)}, \\ \mathscr{R}_1(t) = [(e)\mathscr{I}_0 - \delta\mathscr{R}_0]\frac{t^{\sigma}}{\Gamma(\sigma+1)}, \\ \mathscr{H}_2(t) = [(h_{11})(h_{22})]\frac{t^{2\sigma}}{\Gamma(2\sigma+1)} + [b\mathscr{H}_0 i_{11}]\frac{t^{2\sigma}}{\Gamma(2\sigma+1)}, \\ \mathscr{I}_2(t) = [(i_{22})(i_{33})]\frac{t^{2\sigma}}{\Gamma(2\sigma+1)} + [b\mathscr{I}_0(h_{33})]\frac{t^{2\sigma}}{\Gamma 2\sigma+1}, \\ \mathscr{R}_2(t) = [(e)i_{33}]\frac{t^{2\sigma}}{\Gamma(2\sigma+1)} - [\delta r_{11}]\frac{t^{2\sigma}}{\Gamma(2\sigma+1)} \end{cases} \tag{17}$$

and so on. The rest of terms can be calculated in the same way. The unknown values in (17) are given as

$$\begin{aligned} h_{11} &= a + b\mathscr{I}_0 - \delta, \\ h_{22} &= (a - b\mathscr{I}_0 + c - \delta)\mathscr{H}_0, \\ i_{11} &= b\mathscr{H}_0\mathscr{I}_0 + (c - d - e - \delta)\mathscr{I}_0, \\ i_{22} &= b\mathscr{H}_0 + c - d - e - \delta, \\ i_{33} &= b\mathscr{H}_0\mathscr{I}_0 + (c - d - e - \delta)\mathscr{I}_0, \\ h_{33} &= b\mathscr{I}_0\mathscr{H}_0(a - b\mathscr{I}_0 + c - \delta), \\ r_{11} &= (e)\mathscr{I}_0 - \delta\mathscr{R}_0. \end{aligned}$$

6. Graphical results and discussion

To present the concerned approximate solutions (16) of the model under consideration, we recall some numerical values for the parameters in the given Table 1. **Case-I** In this case we take the values of Table 1 and simulate the three compartments for $t = 300$ days in Figs. 1–3.

On small immigration rate, the behavior of approximate solutions of various compartment has presented in Figs. 1–3. We see that at various arbitrary order the declines in susceptible and infected class is different. The decay is more at small non-integer order while the recovery growth is also high at small fractional order. The concerned values are just assumption to see the behaviors of non-integer order approximate solution of the proposed arbitrary order model. We can say that the results of fractional order derivatives are comparable with that of integer order and fractional calculus can also well describe the situation of dynamical systems.

Case-II: In this case we changing the values of $\delta = 0.019$ and $e = 0.00023$ and the remaining values are the same as given in Table 1 in Figs. 4–6.

Case-III: In the third case we again changes the values of $\delta = 0.00019$ and $e = 0.000023$ while the remaining values are fixed of Table 1.

Table 1 The physical interpretation of the parameter.

Parameters	The physical interpretation	Numerical values
$\mathscr{H}_0$	Susceptible compartment	8.567761 millions [45]
$\mathscr{I}_0$	Infected compartment	0.567261 millions [45]
$\mathscr{R}_0$	Recovered compartment	0.530597 millions [45]
a	Immigration rate of susceptible class	0.0018
b	The infection rate	0.003
c	Immigration rate of infected class	0.005
d	Death rate due to infection	0.019
δ	Natural death rate	0.0019
e	Cure rate from infection	0.00005

7. Numerical simulation by Non-Standard Finite Difference Scheme (NSFD)

In this section we will analyze the considered model for approximate solution using the Grünwald-Letnikov techniques of approximation for the 1-D arbitrary order as given

$$\mathscr{D}^{\sigma}(Y(t)) = \lim_{\Delta\to 0}\Delta^{-\sigma}\sum_{i=0}^{n}(-1)^i\binom{\sigma}{i}Y(t - i\Delta), \tag{18}$$

where $t = n\Delta$ and Δ is interval. Now, we take the problem as

$$\begin{cases} \mathscr{D}^{\sigma}(Y(t)) = g(t, Y(t)), \; t \in [0, T], \; T < \infty, \\ Y(t_0) = Y_0. \end{cases} \tag{19}$$

Now by (18), the left side of (19) is discretized and one may write as

$$\sum_{i=0}^{n} K^{\sigma} Y(t_{p-i}) = g(t, Y(t)), \quad p = 1, 2, 3 \ldots, \tag{20}$$

where $t_p = p\Delta$ and K_i^{σ} are the Grünwald-Letnikov coefficients defined as

$$K_i^{\sigma} = (1 - \frac{1+\sigma}{i})K_{i-1}^{\sigma}, \quad i = 1, 2, 3 \ldots \tag{21}$$

and

$$K_0^{\sigma} = \Delta^{-\sigma}.$$

Next we get the non-standard finite difference scheme (NSFD) by Grünwald-Letnikov method for the arbitrary order model (2) as follows

$$\begin{cases} \mathscr{H}(t_{p+1}) = \frac{1}{K_0^{\sigma}}\left[-\sum_{q=1}^{p+1}K_q^{\sigma}\mathscr{H}(t_{p+1-q}) + \mathscr{H}(t)a - \mathscr{H}(t)b\mathscr{I}(t) + (-\delta)\mathscr{H}(t)\right], \\ \mathscr{I}(t_{p+1}) = \frac{1}{K_0^{\sigma}}\left[-\sum_{q=1}^{p+1}K_q^{\sigma}\mathscr{I}(t_{p+1-q}) + \mathscr{H}(t)b\mathscr{I}(t) + (-d - e + c - \delta)\mathscr{I}(t)\right], \\ \mathscr{R}(t_{p+1}) = \frac{1}{K_0^{\sigma}}\left[-\sum_{q=1}^{p+1}K_q^{\sigma}\mathscr{R}(t_{p+1-q}) + (e)\mathscr{I}(t) - \delta\mathscr{R}(t)\right]. \end{cases} \tag{22}$$

We will now simulate the three compartments of the considered equations by (NSFD) using the same data of Table 1 as in the cases of (LADM) (see Figs. 7–9).

Case-I: In the first case we obtain the simulation by (NFSD) using the data of Table 1 in Figs. 10–12 at different fractional order. Such type of simulation will also provide a comparable results as by (LADM).

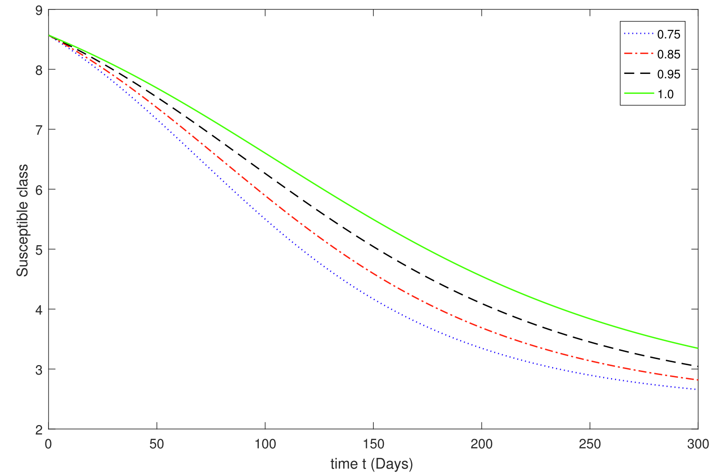

Fig. 1 Graphical representation of series solution for $\mathscr{H}(t)$ corresponding to various arbitrary order of σ.

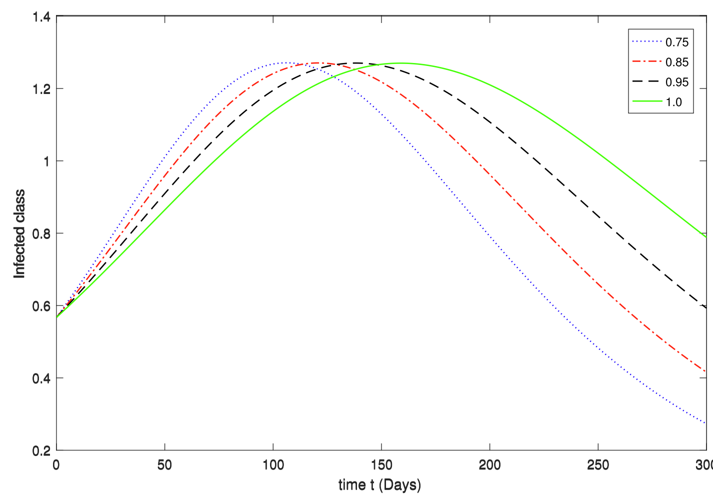

Fig. 2 Graphical representation of series solution for $\mathscr{I}(t)$ corresponding to various arbitrary order of σ.

Case-II: In the second case we obtain the simulation by (NFSD) by changing $\delta = 0.019$ and $e = 0.00023$ while the rest of the data is of Table 1 in Figs. 13–15 at different fractional order.

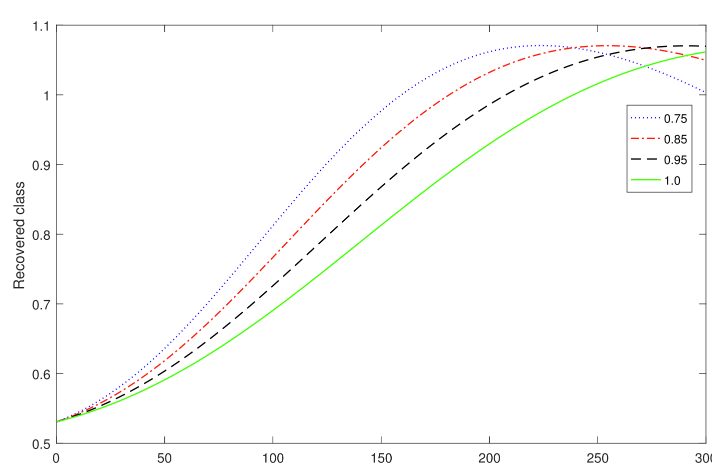

Fig. 3 Graphical representation of series solution for $\mathscr{R}(t)$ corresponding to various arbitrary order of σ.

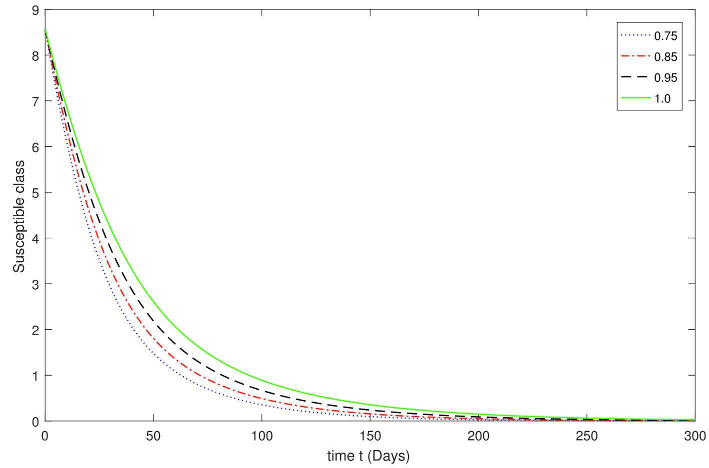

Fig. 4 Graphical representation of series solution for $\mathscr{H}(t)$ corresponding to various arbitrary order of σ.

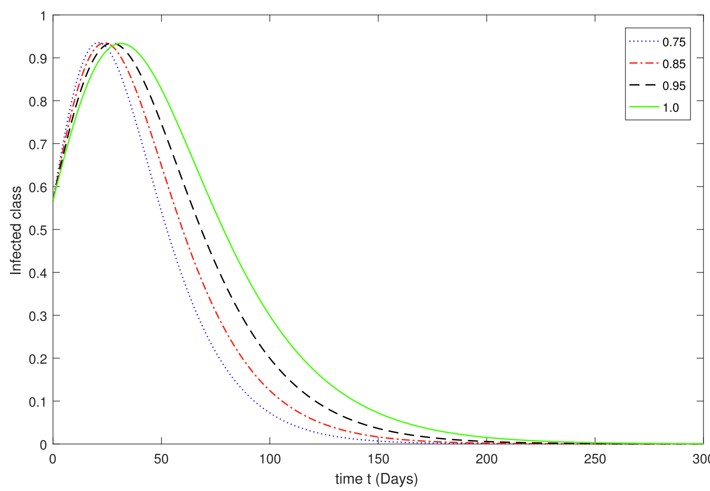

Fig. 5 Graphical representation of series solution for $\mathscr{I}(t)$ corresponding to various arbitrary order of σ.

Case-III: In this case we have drawn the simulation by (NFSD) by changing $\delta = 0.00019$ and $e = 0.000023$ while the rest of the data is of Table 1 in Figs. 16–18 at different fractional order.

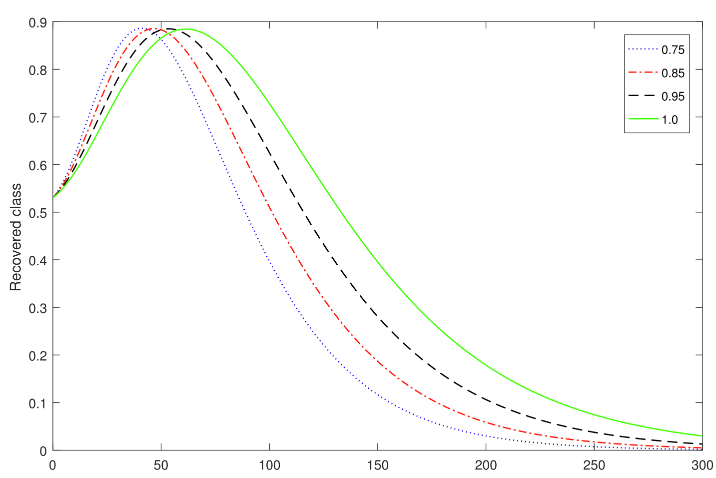

Fig. 6 Graphical representation of series solution for $\mathscr{R}(t)$ corresponding to various arbitrary order of σ.

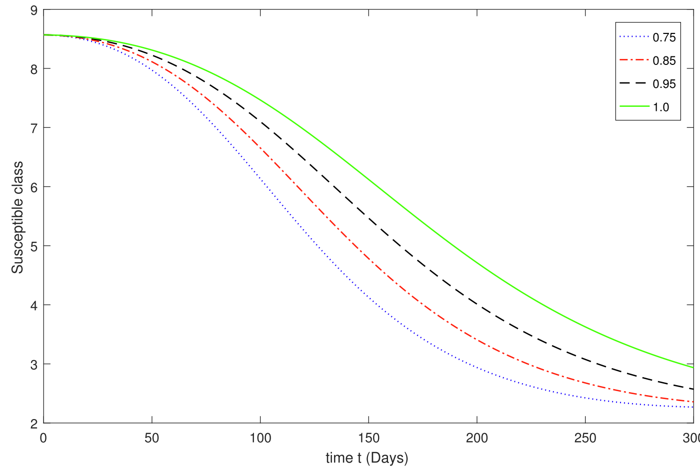

Fig. 7 Graphical representation of series solution for $\mathscr{H}(t)$ corresponding to various arbitrary order of σ.

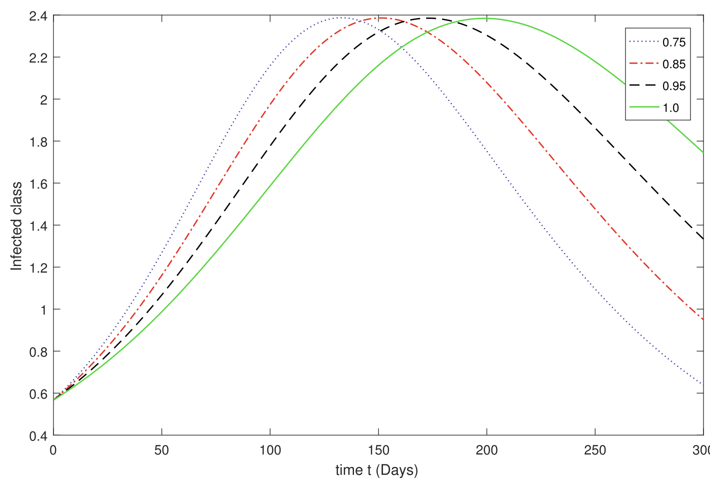

Fig. 8 Graphical representation of series solution for $\mathscr{I}(t)$ corresponding to various arbitrary order of σ.

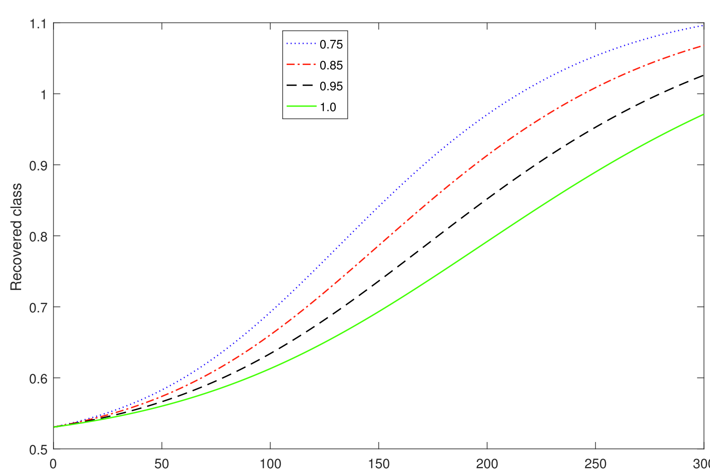

Fig. 9 Graphical representation of series solution for $\mathscr{R}(t)$ corresponding to various arbitrary order of σ.

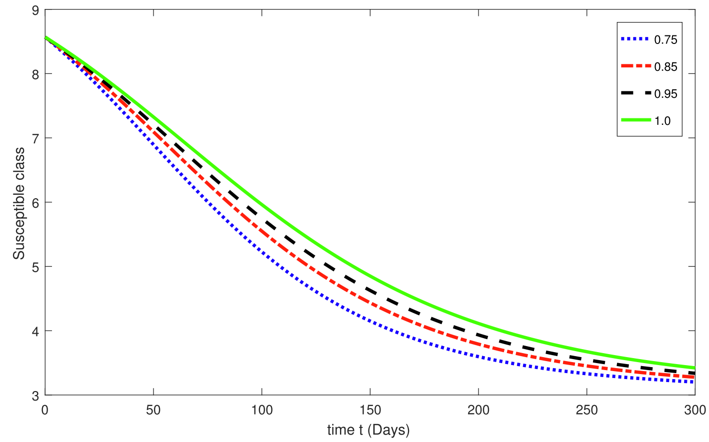

Fig. 10 Graphical representation of numerical solution for $\mathscr{H}(t)$ by (NFSD) corresponding at various arbitrary order of σ.

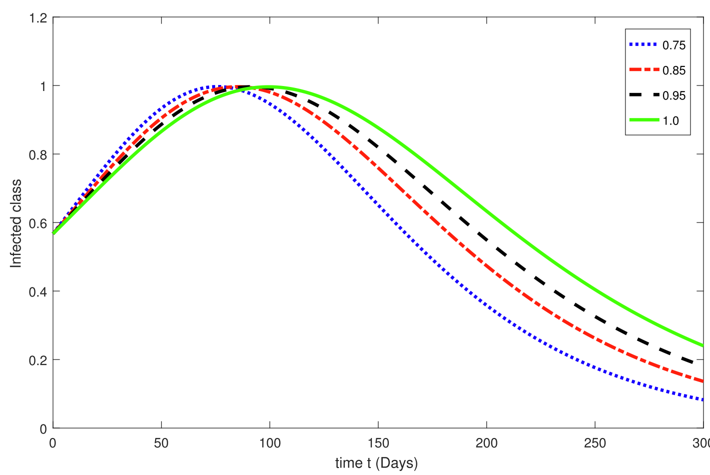

Fig. 11 Graphical representation of numerical solution for $\mathscr{I}(t)$ by (NFSD) corresponding at various arbitrary order of σ.

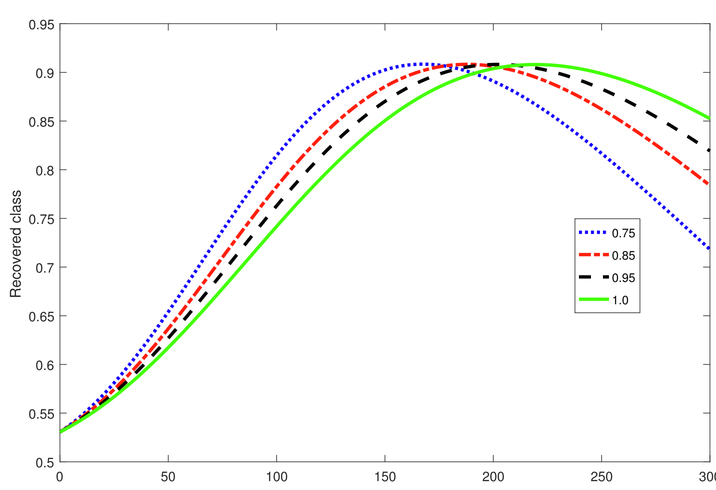

Fig. 12 Graphical representation of numerical solution for $\mathscr{R}(t)$ by (NFSD) corresponding at various arbitrary order of σ.

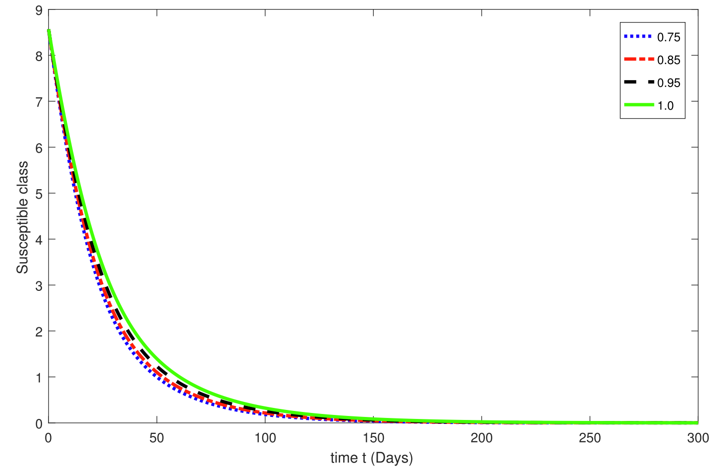

Fig. 13 Graphical representation of numerical solution for $\mathscr{S}(t)$ by (NFSD) corresponding at various arbitrary order of σ.

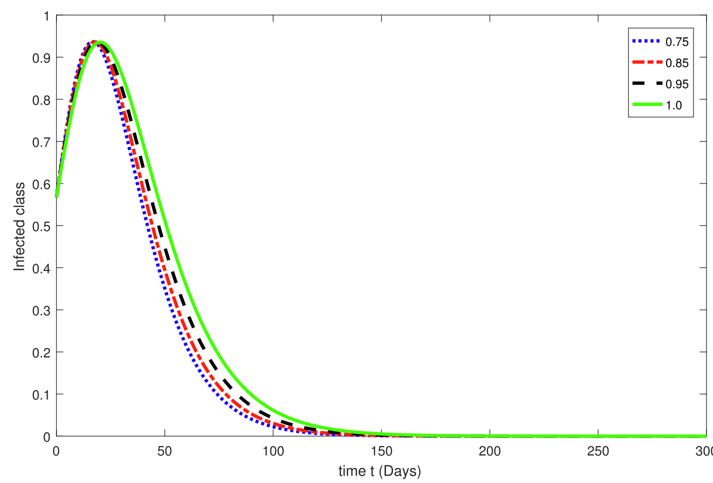

Fig. 14 Graphical representation of numerical solution for $\mathscr{I}(t)$ by (NFSD) corresponding at various arbitrary order of σ.

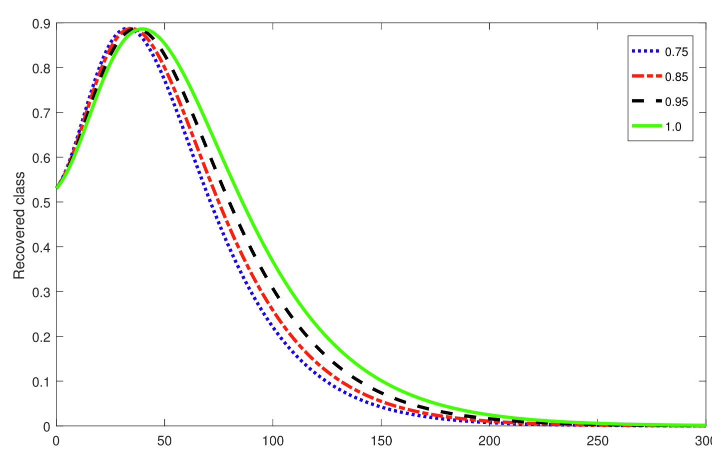

Fig. 15 Graphical representation of numerical solution for $\mathscr{R}(t)$ by (NFSD) corresponding at various arbitrary order of σ.

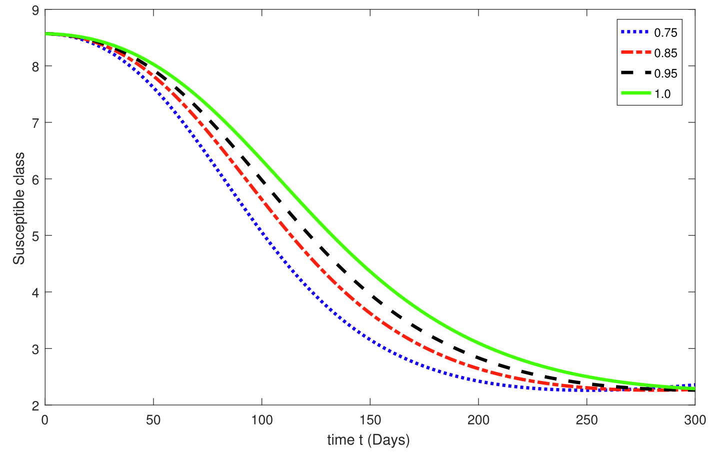

Fig. 16 Graphical representation of numerical solution for $\mathscr{H}(t)$ by (NFSD) corresponding at various arbitrary order of σ.

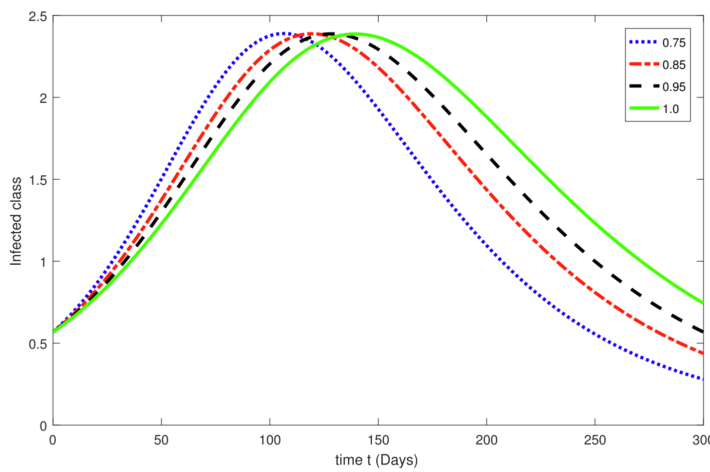

Fig. 17 Graphical representation of numerical solution for $\mathscr{I}(t)$ by (NFSD) corresponding at various arbitrary order of σ.

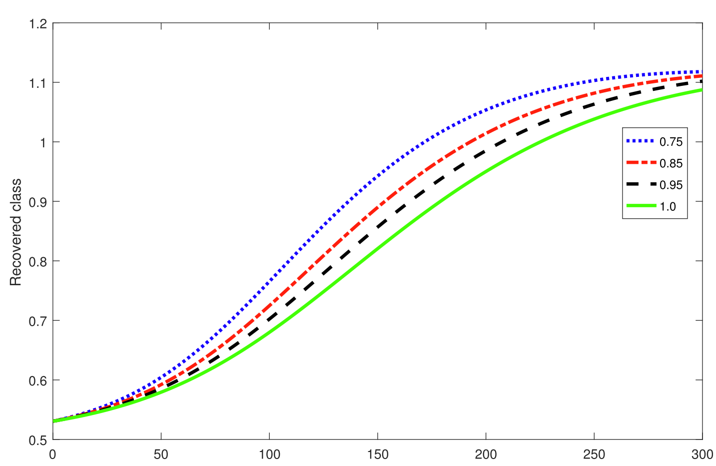

Fig. 18 Graphical representation of numerical solution for $\mathscr{R}(t)$ by (NFSD) corresponding at various arbitrary order of σ.

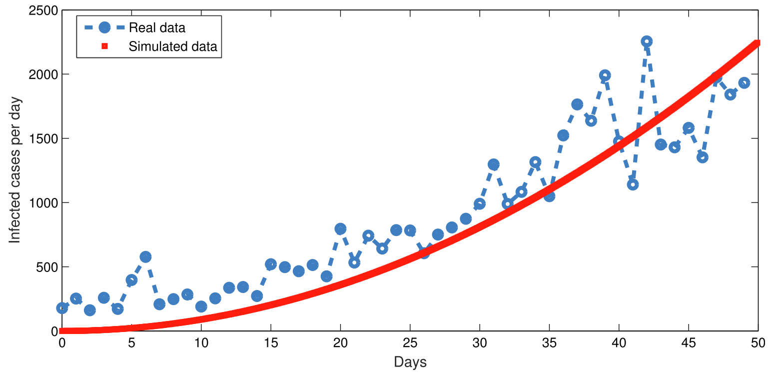

Fig. 19 Comparison between real data and simulated data in case of infection for fifty days.

Here we take some real data of infected cases in Pakistan from First April 2021 to 20the May 2021 as in [46]

$$[178, 252, 161, 258, 172, 397, 577, 208, 248, 284, 190, 254, 336, 342, 272, 520, 497, 465, 514, 425, 796, 533, 742, 642, 785, 783, 605, 751, 806, 874, 990, 1297, 989, 1083, 1083, 1315, 1049, 1523, 1764, 1637, 1991, 1476, 1140, 2255, 1452, 1430, 1581, 1352, 1974, 1841, 1932].$$

in Fig. 19.

8. Concluding remarks

We have investigated a fractional order mathematical model of severe acute respiratory syndrome coronavirus-2 (COVID-19) by using the non-integer order derivative in Caputo sense. The results for boundedness and feasibility of solution have been provided. Some results regarding qualitative analysis of the model have also been proved by using non-linear functional analysis. By using the famous integral transform due to Laplace along with decomposition techniques and Adomian polynomial, we have computed series type solutions for the proposed problem. Taking some different numerical values for natural death rateδ and recovery rate e we have graphically presented three different cases of the solution. Further the Non-Standard Finite Difference Scheme is also applied to the given problem for investigation of iterative approximate solution. The results obtained by LADM are comparable with the results of NSFD at different fractional orders. The simulation converges rapidly at low order lying between 0 and 1 as compared to higher order. We With very small immigration rate of infectious peoples and high protection rate, the current infection may be controlled up to certain extent. So we conclude that arbitrary order derivatives along with the effect some parameter can well explain the mathematical models of real world phenomena along with history information.

Declaration section

Availability of Data and Materials Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

Funding There is no funding source available for this article.

Authors Contribution All authors equally contributed this manuscript and approved the final version.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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